

# Null Melvin Twist to Sakai-Sugimoto model

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## Abstract

The Null Melvin Twist to D4 brane background has been studied and putting in a bunch of coincident D8 and anti-D8 branes *a la* Sakai-Sugimoto model results in a breakdown of a global symmetry to its diagonal subgroup. The same is studied at finite temperature which results in the unchanged form of the chiral symmetry breaking and restoration curve. The effect of turning on a finite chemical potential do not changes the structure of the phase diagram with respect to Null Melvin Twist. Moreover, the confining and de-confining phase transition has the same structure as in the absence of Null Melvin Twist.

# 1 Introduction

It is certainly important and interesting to understand the strongly coupled feature of any field theory but in practice it is not an easy job. However, it can be made easy, in some limit, if we know the corresponding gravity dual. In this context, if we know a system that preserve some symmetry may be relativistic or non-relativistic type, then the way to understand is by the prescription of gauge gravity correspondence, which means one need to find an appropriate geometric description for a system whose dual field theory at strong coupling exhibiting the above symmetry.

In general it is not easy to find such a gravitational description, but given a gravity description which respects that symmetry, it is easy to generate another gravitational description, using the symmetry of the problem. In this context, if we start with a theory that has, let us say, Poincare symmetry then using the solution generating techniques we can have new solutions for which the system can have the same or different symmetry. Here we are interested in the subgroup of the original symmetry group i.e. it possesses the Galilean symmetry. This new solution is obtained by the use of Null Melvin Twist (NMT) or the Discreet Light Cone Quantisation (DLCQ). One of the difference between these two techniques is that in the later case the light cone direction need to be compact, which means the momentum can not be any more continuous, whereas in the earlier case we do not require this to be the case otherwise we won't be able boost along the this direction and the momentum is continuous. Because of this compact nature of the light cone direction, special care need to be required for the zero modes [1], for other important references see [2] and [3] and references therein.

In this paper we shall employ the earlier approach i.e. the null Melvin twist to generate new gravitational solution from the known solution. The effect of NMT is to generate a solution which is of the plane wave type [4] and some aspects of the dual field theory has been studied in [5].

In this context, the solution generating technique [4] has been applied to the super conformal solution of D3 brane and new solutions has been obtained and has been interpreted as the solution showing non-relativistic conformal symmetry [6] and [7]. This solution generating technique is becoming interesting due to the appearance of Galilean (conformal) invariance and is believed to be useful in the understanding of the strongly coupled behaviour of some condensed matter systems. Recently, the explicit form of

a solution has been constructed which preserves this symmetry [8] and [9] and this solution has been embedded in string theory in [3], [6] and [7] using the above mentioned techniques.

There has been several other important, interesting and independent studies by other groups [12],[13],[14],[10], [11] and references therein.

Recently, there appeared another solution [17] in 3+1 dimension, which has an intrinsic exponent  $z$  in a coordinate independent way i.e. the Ricci scalar, square of Ricci tensor  $R_{MN}R^{MN}$  and the square of Riemann tensor  $R_{MNKL}R^{MNKL}$  depends on this exponent  $z$ , whereas the metric written in [8],[9], [7],[6] do depends on this exponent but not the coordinate invariant quantities. Probably, we can say that both of these kind of solutions do not fall in the same class. Another way to see these solutions are not of the same type is that one type of solutions are of plane wave type whereas [17] is not. It is interesting to note that for  $z=1$  both of these kind of solutions satisfy the relation  $R_{MN} \sim g_{MN}$ , whereas as soon as we go away from  $z=1$  then none of these solutions preserves the relation  $R_{MN} \sim g_{MN}$ . Which we may interpret as that both of these class of solutions are non-maximally symmetric space times for  $z$  away from 1. We can generalize the solution of [17] by introducing another exponent and study the correlation function [18].

The NMT has already been applied to systems for odd value of  $p$  of  $D_p$  brane which are charged under electric field in [19] and some other systems using different techniques in [20]-[23] and some other aspects of the plane wave solutions has been studied in [24]-[31] and references therein.

Here we are interested to study a daughter field theory via gauge gravity duality whose parent field theory exhibits some interesting properties, namely, shows confinement and upon inclusion of flavors it shows chiral symmetry breaking and at finite temperature the system restores the chiral symmetry.

The study of strongly coupled field theory via AdS/CFT correspondence has attracted a lot interest and attention. In this context, the obvious thing to try and understand is the study of the strongly coupled behavior of QCD, but to understand QCD via the above mentioned correspondence means we are studying it in the large but fixed 't Hooft coupling and at large rank of the gauge group. More importantly, finding a suitable geometric description of QCD has been very non-trivial. Moreover, QCD contains flavor degrees of freedom, which transforms fundamentally under the color gauge group, it means from the holographic dual bulk theory point of view, we need to

include flavors and there exists an approach to incorporate and fix the arbitrariness of the number of flavors.

One such approach is called the probe-brane approximation, where one considers the limit in which the number of branes that produces the gluonic degrees of freedom, called color branes, are large in comparison to the number of flavors [33]. It means the flavor branes do not back react on the color branes and hence the presence of flavor branes are not going to change the background geometry generated by the color brane. In order to understand the physics of these flavors more accurately, one needs to consider the full back reaction of the flavor branes onto the color brane.

The studies to understand several properties of QCD till now is either by the top-down [39] or the bottom-up approaches [34], [35] and [36]. In the former case one starts with a theory which has got either the right color gauge group, matter content and couplings or it shares some features with QCD. One such example is the Sakai-Sugimoto model, which displays confinement and chiral symmetry breaking, and this particular approach has got its own problem, like one has to deal here via the probe brane approximation, an unwanted  $SO(5)$  symmetry group, and the most troublesome complication is the presence of KK modes<sup>1</sup>. However, it is not the case with the bottom-up approach, where one starts with a theory with a minimal matter content and the gauge group and assumes that it has the correct coupling and then proceeds to calculate several interesting quantities like chiral symmetry breaking and current-current correlators [34] using AdS/CFT correspondence.

From the days of AdS/QCD, several interesting studies have been made to understand the holographic QCD better. In this context, the IR properties like chiral symmetry breaking, its restoration, confinement and the transition between the confining and de-confining phases have been understood [41]-[46] from the top-down approach, than its UV properties like the asymptotic freedom [35], which is studied using the bottom-up approach. Since these are the most important properties of QCD, means we should have them when we construct any model holographically, at least it should fall in the same class of QCD by showing these properties. Until, now it has not been possible to embed all these properties into a single model from top-down approach. The

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<sup>1</sup>Some studies have been made in [38] using the solution generating technique [37], but more interesting studies need to be made.

vev of the bi-fundamental field associated to the chiral symmetry breaking is studied in [47], [48] and [52] using open Wilson line and tachyon condensation in [49], [50] and [51].

In this paper, we would like to understand better the Sakai-Sugimoto model. In particular, applying the Null Melvin Twist (NMT) to the near horizon limit of D4 brane solution, generates a solution that asymptotes to a plane wave solutions and would study different aspects of Sakai-Sugimoto model like: how does this deformation to the original solution changes confining de-confining transition and also the chiral symmetry breaking and restoration curve?

The results of this study can be summarized as:

(1) The application of NMT to the near extremal solution breaks the rotational symmetry  $SO(5)$  transverse to the D4 brane. However, it does not break the  $SO(5)$  symmetry of the zero temperature solution.

(2) Even though there is no rotation in the NMT resulted solution but there appears the angular dependence in the metric component parallel to the brane direction. The consequence of this is that the scalar field moving in this geometry is not anymore just a function of the radial coordinate at zero frequency and zero momentum, which makes the computation of the correlation functions very complicated.

(3) As a result the coefficient of shear viscosity is not any more given by the formula eq(3.6) of hep-th/0309213.

(4) The NMT resulted solution to the D4 brane has got the same four-form field strength for both zero and non-zero temperature solution. The dilaton for the non-zero temperature is different whereas at zero temperature it remains same.

(5) The periodicity associated to Euclidean time circle for the black hole solution and the  $\tau$  circle for the zero temperature solution are same as before the application of NMT.

(6) The curve that describes the chiral symmetry breaking and restoration [42],[46] remains unchanged.

(7) The curve for confining and de-confining transition [42],[46] also remain unchanged.

## 2 The D4 brane solution

Let us consider a model that talks about generating holographically a gravity solution which may fall in the same class as that of QCD as far as only confinement and chiral symmetry breaking is concerned. The model is based on the D4 brane solution with no supersymmetry, in string frame it reads

$$\begin{aligned} ds_s^2 &= \left(\frac{r}{R}\right)^{3/2}[-dt^2 + \sum_1^3 dx^i dx_i + f(r)d\tau^2] + \left(\frac{R}{r}\right)^{3/2}[dr^2 + r^d \Omega_4^2], \\ \Phi &= \text{Log}[g_s \left(\frac{r}{R}\right)^{3/4}], \quad F_4 = \frac{2\pi N_c}{v_4} \epsilon_4, \quad f(r) = 1 - \left(\frac{r_0}{r}\right)^3 \end{aligned} \quad (1)$$

where  $v_4 = \frac{8\pi^2}{3}$ ,  $\epsilon_4$  is the volume form of the unit  $S^4$  and  $R^3 = \pi g_s N_c l_s^3$ . Note that only this form of normalization of the  $F_4$  flux satisfy the necessary flux quantization condition in units of  $2\pi l_s = 1$ , which is

$$\int_{S^4} \star_{10} F_6 = \int_{S^4} F_4 = 2\pi N_c \quad (2)$$

and if we do not work in any such units then  $v_4 = \frac{2\pi}{3l_s^3}$  and the normalization condition is

$$\int_{S^4} \star_{10} F_6 = \int_{S^4} F_4 = (4\pi^2 \alpha')^{\frac{3}{2}} N_c \quad (3)$$

This background is reliable for a certain range of energy scale [40], for which the effective coupling in field theory should be very large or in other words the Ricci scalar should be very small in units of string length,  $l_s$ . To suppress the effects of string loop corrections, we need to take small values to dilaton,  $e^\Phi \ll 1$ .

According to gauge gravity correspondence, the five dimensional coupling is related to a dimensionless effective coupling  $g_{eff}$  [40]

$$g_{eff}^2 = g_5^2 N_c U, \quad \text{with} \quad g_5^2 = (2\pi)^2 g_s l_s, \quad (4)$$

where  $U = \frac{r}{\alpha'}$  and the four dimensional 't Hooft coupling is related to the five dimensional coupling

$$g_{YM}^2 N_c = \frac{g_5^2 N_c}{2\pi R_\tau}, \quad (5)$$

where  $2\pi R_\tau$  is the periodicity of the  $\tau$  circle and is written in eq(55). It follows that the four dimensional 't Hooft coupling is  $g_{YM}^2 N_c = \frac{3}{l_s^2} R^{3/2} \sqrt{r_0}$ . From the D4 solution, the Ricci scalar gets a maximum value at  $r = (\frac{7}{5})^{1/3} r_0$  and is

$$R_s = -\frac{135}{14} \frac{(5/7)^{1/6}}{R^{3/2} \sqrt{r_0}} \quad (6)$$

Demanding that  $\alpha' |R_s| \ll 1$  gives

$$\frac{R^{3/2} \sqrt{r_0}}{\alpha'} \gg 1. \quad (7)$$

which means the four dimensional 't Hooft coupling is large i.e.  $g_{YM}^2 N_c \gg 1$ . So, roughly, its the energy that stay close to IR makes the gravity description reliable.

From the dilaton constraint we get

$$U \ll \frac{4\pi^{7/3} N^{4/3}}{g_{YM}^2 N_c 2\pi R_\tau} = \frac{3\pi^{1/3} N^{1/3}}{g_s l_s^3} \quad (8)$$

### 3 Null Melvin Twist

We would like to do Null Melvin Twist (NMT) to backgrounds of the form

$$ds^2 = A(r)dt^2 + B(r)dy^2 + \delta_{ij}(r)dx^i dx^j + C(r)dr^2 + D(r)d\Omega_4^2 \quad (9)$$

There could be other fields present that are coming either from the NS-NS or RR sector or both. As a particular example, we shall stick to the D4 brane geometry, for which there are not any other fields coming from the NS-NS sector apart from dilaton. The dilaton

$$\Phi(r) = \Phi_0(r) \quad (10)$$

and the 4-form flux is taken as

$$F_4 = f_0 \epsilon_4, \quad (11)$$

where  $f_0$  is some normalization constant and  $\epsilon_4$  is the volume form of the 4-sphere.

Later we shall decompose the metric of  $d\Omega^2$  explicitly when we do the twist along one of its isometry direction.

The proposal of doing Null Melvin twist consists of six steps [4].

1. Boost along a translationally invariant direction,  $y$ , by an amount  $\gamma$ .
2. T-dualize along this direction,  $y$ .
3. Twist a one form i.e. do rotations in different planes
4. T-dualize again on  $y$ .
5. Boost by  $-\gamma$  along the same direction,  $y$
6. Take a specific limit in which  $\alpha \rightarrow 0$ ,  $\gamma \rightarrow \infty$  and keeping  $\beta = \frac{1}{2}\alpha e^\gamma =$  fixed

By going through these 6 steps, we generate a new background and in the  $\beta \rightarrow 0$ , we get back our original solution.

Notation:  $c = \cosh \gamma$ ,  $s = \sinh \gamma$

Let us start to do the NMT for the geometry written above<sup>2</sup>.

Step 1:

$$dt \rightarrow cdt - sdy, \quad dy \rightarrow -sdt + cdy \quad (12)$$

$$\begin{aligned} ds^2 &= (Ac^2 + Bs^2)dt^2 + (As^2 + Bc^2)dy^2 - 2cs(A+B)dtdy + \\ &\quad \delta_{ij}(r)dx^i dx^j + C(r)dr^2 + D(r)d\Omega_4^2, \\ \Phi(r) &= \Phi_0(r), \quad F_4 = f_0\epsilon_4. \end{aligned} \quad (13)$$

Step 2:

$$\begin{aligned} ds^2 &= \frac{AB}{(As^2 + Bc^2)}dt^2 + \frac{dy^2}{(As^2 + Bc^2)} + \delta_{ij}(r)dx^i dx^j + C(r)dr^2 + D(r)d\Omega_4^2, \\ \Phi(r) &= \Phi_0(r) - \frac{1}{2}\text{Log}[(As^2 + Bc^2)], \quad B_{ty} = -\frac{cs(A+B)}{(As^2 + Bc^2)}, \\ F_5 &= f_0 dy \wedge \epsilon_4 \end{aligned} \quad (14)$$

In order to perform step 3, let us write down the metric of unit 4-sphere as

$$d\Omega_4^2 = dx_1^2 + \dots + dx_5^2, \quad (15)$$

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<sup>2</sup>Sometimes we write  $A(r)$  and  $B(r)$  etc, as simply  $A$  and  $B$  etc, just to avoid cluttering of brackets.



with the restriction  $x_1^2 + \dots + x_5^2 = 1$ . The twisting is done following [4]

$$\begin{aligned} x_1 + ix_2 &\rightarrow e^{i\alpha y}(x_1 + ix_2) \\ x_3 + ix_4 &\rightarrow e^{i\alpha y}(x_3 + ix_4), \end{aligned} \quad (16)$$

where we have done equal rotation in both the planes. This gives the resulting unit 4-sphere metric

$$d\Omega_4^2 \rightarrow d\Omega_4^2 + \alpha\sigma dy + \alpha^2(1 - x_5^2)dy^2, \quad (17)$$

where

$$\frac{\sigma}{2} = x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3. \quad (18)$$

A specific realization of unit  $S^4$  could be

$$\begin{aligned} x_1 &= \sin\theta \sin\varphi \sin\psi, & x_2 &= \sin\theta \sin\varphi \cos\psi \\ x_3 &= \sin\theta \cos\varphi \sin\chi, & x_4 &= \sin\theta \cos\varphi \cos\chi \\ x_5 &= \cos\theta, \end{aligned} \quad (19)$$

with the ranges for the angles are  $0 \leq \theta \leq \pi$ ,  $0 \leq \varphi \leq \frac{\pi}{2}$ ,  $0 \leq \psi, \chi \leq 2\pi$ .

It means the resulting unit 4-sphere metric with this realization becomes

$$d\Omega_4^2 \rightarrow d\Omega_4^2 + \alpha\sigma dy + \alpha^2 \sin^2\theta dy^2 \quad (20)$$

The one form

$$\frac{\sigma}{2} = -\sin^2\theta \sin^2\varphi d\psi - \sin^2\theta \cos^2\varphi d\chi \quad (21)$$

and the

$$d\Omega_4^2 = d\theta^2 + \sin^2\theta d\Omega_3^2, \quad (22)$$

where

$$d\Omega_3^2 = d\varphi^2 + \sin^2\varphi d\psi^2 + \cos^2\varphi d\chi^2 \quad (23)$$

Due to this twisting the volume form  $\epsilon_4$  will be different. This can be seen as follows.

The metric of the unit 4-sphere can be re-written as

$$\begin{aligned}
ds_4^2 = & \frac{1}{1 - \sum_1^4 x_i^2} \left( (1 - \sum_1^4 x_i^2 + x_1^2) dx_1^2 + (1 - \sum_1^4 x_i^2 + x_2^2) dx_2^2 + \right. \\
& (1 - \sum_1^4 x_i^2 + x_3^2) dx_3^2 + (1 - \sum_1^4 x_i^2 + x_4^2) dx_4^2 + 2x_1x_2 dx_1 dx_2 + 2x_1x_3 dx_1 dx_3 + \\
& \left. 2x_1x_4 dx_1 dx_4 + 2x_2x_3 dx_2 dx_3 + 2x_2x_4 dx_2 dx_4 + 2x_3x_4 dx_3 dx_4 \right) \quad (24)
\end{aligned}$$

The volume form of this metric is

$$\epsilon_4 = \frac{1}{\sqrt{1 - \sum_1^4 x_i^2}} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4, \quad (25)$$

expressing this in terms of the angular coordinates give

$$\epsilon_4 = \sin^3 \theta \sin \varphi \cos \varphi d\theta \wedge d\varphi \wedge d\psi \wedge d\chi \quad (26)$$

Now using the following fact

$$\begin{aligned}
dx_1 + idx_2 & \rightarrow e^{i\alpha y} [(dx_1 + idx_2) + i\alpha dy(x_1 + ix_2)] \\
dx_3 + idx_4 & \rightarrow e^{i\alpha y} [(dx_3 + idx_4) + i\alpha dy(x_3 + ix_4)] \quad (27)
\end{aligned}$$

we get

$$\begin{aligned}
dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 & \rightarrow dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 - \\
& \alpha dy \wedge [(x_3 dx_3 + x_4 dx_4) \wedge dx_1 \wedge dx_2 + (x_1 dx_1 + x_2 dx_2) \wedge dx_3 \wedge dx_4] \quad (28)
\end{aligned}$$

This means the volume form changes under the twisting to

$$\epsilon_4 \rightarrow \epsilon_4 - \alpha \sin^3 \theta \sin \varphi \cos \varphi dy \wedge d\theta \wedge d\varphi \wedge d(\chi - \psi) \quad (29)$$

It means under this twisting the five form flux generated after step 2, do not gets changed.

Step 3: After the twist the solution looks as

$$\begin{aligned}
ds^2 &= \frac{AB}{(As^2 + Bc^2)} dt^2 + \frac{dy^2}{(As^2 + Bc^2)} + \delta_{ij}(r) dx^i dx^j + Cdr^2 + Dd\Omega_4^2 + \\
&D\alpha\sigma dy + D\alpha^2 \sin^2\theta dy^2, \Phi(r) = \Phi_0(r) - \frac{1}{2} \text{Log}[(As^2 + Bc^2)], \\
B_{ty} &= -\frac{cs(A+B)}{(As^2 + Bc^2)}, F_5 = f_0 dy \wedge \epsilon_4, \\
\Phi(r) &= \Phi_0(r) - \frac{1}{2} \text{Log}[(As^2 + Bc^2)]
\end{aligned} \tag{30}$$

Step 4:

T-dualizing the solution along y direction that resulted after step 3 gives

$$\begin{aligned}
ds^2 &= Xdt^2 + Wdy^2 + 2Zdtdy + \delta_{ij}(r) dx^i dx^j + Cdr^2 + Dd\theta^2 + D\sin^2\theta d\varphi^2 + \\
&Ld\psi^2 + Md\chi^2 + 2Nd\chi d\psi,
\end{aligned} \tag{31}$$

$$\Phi = \Phi_0 - \frac{1}{2} \text{Log}[K], F_4 = f_0 \epsilon_4$$

$$\begin{aligned}
B &= -\frac{D\alpha\sin^2\theta}{K} [\sin^2\varphi(As^2 + Bc^2) d\psi \wedge dy + \cos^2\varphi(As^2 + Bc^2) d\chi \wedge dy + \\
&\sin^2\varphi cs(A+B) dt \wedge d\psi + \cos^2\varphi cs(A+B) dt \wedge d\chi],
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
K &= 1 + D\alpha^2 \sin^2\theta(As^2 + Bc^2), \\
X &= \frac{ABK + c^2 s^2 (A+B)^2}{K(As^2 + Bc^2)}, \\
W &= \frac{As^2 + Bc^2}{K}, \quad Z = -\frac{cs(A+B)}{K}, \\
L &= \frac{D\sin^2\theta \sin^2\varphi [1 + D\alpha^2 \sin^2\theta \cos^2\varphi(As^2 + Bc^2)]}{K}, \\
M &= \frac{D\sin^2\theta \cos^2\varphi [1 + D\alpha^2 \sin^2\theta \sin^2\varphi(As^2 + Bc^2)]}{K}, \\
N &= -\frac{D^2\alpha^2 \sin^4\theta \sin^2\varphi \cos^2\varphi}{K} (As^2 + Bc^2)
\end{aligned} \tag{33}$$

Step 5: Boosting the solution after step 4 by an amount  $-\gamma$  gives

$$ds^2 = (c^2 X + s^2 W + 2csZ) dt^2 + (s^2 X + c^2 W + 2csZ) dy^2 +$$

$$\begin{aligned}
& 2dtdy[cs(X+W) + (c^2 + s^2)Z] + \delta_{ij}(r)dx^i dx^j + Cdr^2 + Dd\theta^2 + \\
& D\sin^2\theta d\varphi^2 + Ld\psi^2 + Md\chi^2 + 2Nd\chi d\psi, \\
\Phi &= \Phi_0 - \frac{1}{2}\text{Log}[K], \quad F_4 = f_0\epsilon_4 \\
B &= \frac{D\alpha\sin^2\theta}{K}[-\sin^2\varphi Asdt \wedge d\psi + B\sin^2\varphi dy \wedge d\psi - \cos^2\varphi Asdt \wedge d\chi \\
&+ \cos^2\varphi cBdy \wedge d\chi] \tag{34}
\end{aligned}$$

Step 6:

The final step is to take a proper limit for which  $\alpha \rightarrow 0$  and  $\gamma \rightarrow \infty$  with  $\beta = \frac{\alpha}{2}e^\gamma$  fixed, gives

$$\begin{aligned}
ds^2 &= \frac{A}{K}(1 + \beta^2 DB\sin^2\theta)dt^2 + \frac{B}{K}(1 + \beta^2 DAsin^2\theta)dy^2 + 2\frac{ABD\beta^2\sin^2\theta}{K}dtdy \\
&+ \delta_{ij}(r)dx^i dx^j + Cdr^2 + Dd\theta^2 + D\sin^2\theta d\varphi^2 \\
&+ \frac{D}{K}\sin^2\theta\sin^2\varphi(1 + \beta^2 D\sin^2\theta\cos^2\varphi(A+B))d\psi^2 \\
&+ \frac{D}{K}\sin^2\theta\cos^2\varphi(1 + \beta^2 D\sin^2\theta\sin^2\varphi(A+B))d\chi^2 \\
&- 2\frac{\beta^2 D^2\sin^4\theta\sin^2\varphi\cos^2\varphi(A+B)}{K}d\chi d\psi, \\
B_2 &= \frac{D\beta\sin^2\theta}{K}[-\sin^2\varphi Adt \wedge d\psi + B\sin^2\varphi dy \wedge d\psi - \cos^2\varphi Adt \wedge d\chi \\
&+ \cos^2\varphi cBdy \wedge d\chi], \\
\Phi &= \Phi_0 - \frac{1}{2}\text{Log}[K], \quad F_4 = f_0\epsilon_4, \quad K = 1 + \beta^2 D\sin^2\theta(A+B) \tag{35}
\end{aligned}$$

The simplified form of the NMT resulted D4 brane geometry can be written as

$$\begin{aligned}
ds^2 &= \frac{A}{K}dt^2 + \frac{B}{K}dy^2 + \frac{ABD\beta^2\sin^2\theta}{K}(dt + dy)^2 + \delta_{ij}(r)dx^i dx^j + Cdr^2 \\
&+ Dd\theta^2 + D\sin^2\theta d\varphi^2 + \frac{D}{K}\sin^2\theta\sin^2\varphi d\psi^2 + \frac{D}{K}\sin^2\theta\cos^2\varphi d\chi^2 \\
&+ \frac{\beta^2}{K}D^2\sin^4\theta\sin^2\varphi\cos^2\varphi(A+B)(d\psi - d\chi)^2 \tag{36}
\end{aligned}$$

Upon doing the following change of coordinates,  $t = \frac{1}{2}(\tilde{t} - \xi)$ ,  $y = \frac{1}{2}(\tilde{t} + \xi)$ , we can rewrite the metric in a form that shows explicitly the pp-wave structure.

$$\begin{aligned}
ds^2 &= \left( \frac{A+B}{4K} + \beta^2 \frac{ADB}{K} \sin^2 \theta \right) d\tilde{t}^2 + \frac{A+B}{4K} d\xi^2 + 2 \frac{B-A}{4K} d\tilde{t} d\xi + \delta_{ij}(r) dx^i dx^j \\
&+ C dr^2 + D d\theta^2 + D \sin^2 \theta d\varphi^2 + \frac{D}{K} \sin^2 \theta \sin^2 \varphi d\psi^2 + \frac{D}{K} \sin^2 \theta \cos^2 \varphi d\chi^2 \\
&+ \frac{\beta^2}{K} D^2 \sin^4 \theta \sin^2 \varphi \cos^2 \varphi (A+B) (d\psi - d\chi)^2, \\
B_2 &= \frac{D\beta \sin^2 \theta}{2K} [-A(d\tilde{t} - d\xi) + B(d\tilde{t} + d\xi)] \wedge [\sin^2 \varphi d\psi + \cos^2 \varphi d\chi] \quad (37)
\end{aligned}$$

For the example at hand the functions are

$$\begin{aligned}
A &= -f(r) \left( \frac{r}{R} \right)^{3/2}, \quad B(r) = \left( \frac{r}{R} \right)^{3/2}, \quad \delta_{ij}(r) = \delta_{ij} \left( \frac{r}{R} \right)^{3/2}, \quad C(r) = \left( \frac{R}{r} \right)^{3/2} \frac{1}{f(r)}, \\
D(r) &= R^{3/2} \sqrt{r}, \quad f(r) = 1 - \left( \frac{r_T}{r} \right)^3, \quad \Phi_0 = \text{Log}[g_s \left( \frac{r}{R} \right)^{3/4}], \quad f_0 = \frac{2\pi N_c}{v_4}, \quad (38)
\end{aligned}$$

which describes a black hole and  $v_4$  is the unit volume of  $S^4$ , whose value is  $v_4 = \frac{8\pi^2}{3}$ . Re-writing the solution in a better looking way

$$\begin{aligned}
ds^2 &= \frac{1}{K} \left( \frac{r}{R} \right)^{3/2} \left[ -f(1 + r^2 \beta^2 \sin^2 \theta) dt^2 + (1 - f r^2 \beta^2 \sin^2 \theta) dy^2 - 2 f r^2 \beta^2 \sin^2 \theta dt dy \right. \\
&+ K(dx^2 + dz^2 + d\tau^2) \left. \right] + \left( \frac{R}{r} \right)^{3/2} \frac{dr^2}{f} + \frac{\beta^2 R^{3/2} r_0^3}{K \sqrt{r}} \sin^4 \theta \sin^2 \varphi \cos^2 \varphi (d\psi - d\chi)^2 \\
&+ \frac{R^{3/2} \sqrt{r}}{K} \left[ K(d\theta^2 + \sin^2 \theta d\varphi^2) + \sin^2 \theta (\sin^2 \varphi d\psi^2 + \cos^2 \varphi d\chi^2) \right] \\
B_2 &= \frac{\beta r^2}{K} \sin^2 \theta (f dt + dy) \wedge [\sin^2 \varphi d\psi + \cos^2 \varphi d\chi], \\
\Phi &= \Phi_0 - \frac{1}{2} \text{Log}[K], \quad F_4 = f_0 \epsilon_4, \quad K = 1 + \frac{r_T^3}{r} \beta^2 \sin^2 \theta \quad (39)
\end{aligned}$$

In this case it follows that the original  $\text{SO}(5)$  symmetry transverse to the D4 brane is broken to  $\text{SO}(3) \times \text{U}(1)^2$ . The  $\text{SO}(3)$  is spanned by the  $\theta$ ,  $\varphi$  coordinates and the two  $\text{U}(1)$ 's follow from  $\chi$  and  $\psi$  coordinates.

Let us do a double Wick rotation to the solution that is presented in eq(9), then the resulting solution shows a very interesting property that is confinement.

If the functions are

$$\begin{aligned} A &= -\left(\frac{r}{R}\right)^{3/2}, \quad B(r) = \left(\frac{r}{R}\right)^{3/2}, \quad \delta_{11}(r) = \delta_{22}(r) = \left(\frac{r}{R}\right)^{3/2}, \quad \delta_{33}(r) = f(r)\left(\frac{r}{R}\right)^{3/2}, \\ C(r) &= \left(\frac{R}{r}\right)^{3/2} \frac{1}{f(r)}, \quad D(r) = R^{3/2} \sqrt{r}, \quad f(r) = 1 - \left(\frac{r_0}{r}\right)^3, \quad \Phi_0 = \text{Log}[g_s \left(\frac{r}{R}\right)^{3/4}], \\ f_0 &= \frac{2\pi N_c}{v_4}, \end{aligned} \tag{40}$$

then it describes a system at zero temperature which shows confinement for  $\beta = 0$ . Note that for non-zero  $\beta = \pm \frac{\text{cosec } \theta_0}{r_0}$ , the surface  $(r, \theta) = (r_0, \theta_0)$  shows that the  $g_{yy}$  goes to zero. This is just an artifact of the coordinate system we used.

Note in this case  $A + B = 0$ , which means the function  $K = 1$  and the solution can be written in a better way as

$$\begin{aligned} ds^2 &= \left(\frac{r}{R}\right)^{3/2} \left[ - (1 + r^2 \beta^2 \sin^2 \theta) dt^2 + (1 - r^2 \beta^2 \sin^2 \theta) dy^2 - 2r^2 \beta^2 \sin^2 \theta dt dy + \right. \\ &\quad \left. (dx^2 + dz^2 + f d\tau^2) \right] + \left(\frac{R}{r}\right)^{3/2} \frac{dr^2}{f(r)} + R^{3/2} \sqrt{r} d\Omega_4^2, \\ B_2 &= r^2 \beta \sin^2 \theta (dt + dy) \wedge [\sin^2 \varphi d\psi + \cos^2 \varphi d\chi], \\ \Phi &= \Phi_0, \quad F_4 = f_0 \epsilon_4. \end{aligned} \tag{41}$$

In this case the symmetry transverse to the D4 brane is not broken by the NMT twist that is the  $\text{SO}(5)$ . It is very easy to conclude that eq(41) cannot be generated from eq(39) by a double Wick rotation and it implies that NMT do not commute with Wick rotation.

Let us do some change of coordinates  $t = \frac{1}{2}(\tilde{t} - \xi)$ ,  $y = \frac{1}{2}(\tilde{t} + \xi)$  and  $\rho = 2 \frac{R^{3/2}}{\sqrt{r}}$ , using these we can rewrite the solution eq(41)

$$\begin{aligned} ds^2 &= ds_5^2 + \left(\frac{r}{R}\right)^{3/2} f d\tau^2 + R^{3/2} \sqrt{r} d\Omega_4^2, \quad \Phi = \Phi_0, \quad F_4 = f_0 \epsilon_4, \\ B_2 &= 4\sqrt{2} \beta R^3 \frac{\sin^2 \theta}{\rho^2} d\tilde{t} \wedge [\sin^2 \varphi d\psi + \cos^2 \varphi d\chi], \end{aligned} \tag{42}$$

with the

$$ds_5^2 = \tilde{R}^2 \left[ \frac{-2\tilde{\beta}^2 \mathcal{S}(\theta) d\tilde{t}^2}{\rho^{2z+\nu}} + \frac{2d\tilde{t}d\xi + dx_i dx_i + d\rho^2}{\rho^{2+\nu}} \right], \quad (43)$$

where

$$z = 3, \quad \nu = 1, \quad \mathcal{S}(\theta) = \sin^2 \theta, \quad \tilde{R}^2 = 8R^3, \quad \tilde{\beta}^2 = 2^4 R^6 \beta^2. \quad (44)$$

Now we have got two “exponents”  $z$  and  $\nu$  along with a function  $\mathcal{S}(\theta)$ .  
As a comparison with an extremal D3 brane [9]

$$z = 2, \quad \nu = 0, \quad \mathcal{S}(\theta) = 1. \quad (45)$$

Now we may say that the field theory living in 3+1 dimension possessing the Galilean symmetry should have a five dimensional geometry

$$ds_5^2 = \tilde{R}^2 \left[ \frac{-2\tilde{\beta}^2 \mathcal{S}(\theta) d\tilde{t}^2}{\rho^{2z+\nu}} + \frac{2d\tilde{t}d\xi + dx_i dx_i + d\rho^2}{\rho^{2+\nu}} \right] \quad (46)$$

The exponents  $\nu$  and  $z$  are both “dynamical” in the sense that as it depends on the dimensionality of the brane. Hopefully, it will appear in the realistic sense i.e. in the result of the correlation function

$$z = \frac{7-p}{5-p}, \quad \nu = \frac{p-3}{5-p}. \quad (47)$$

This is true for any Dp brane with an exception to D5 brane.

Summarizing, the above analysis it follows that the more feasible looking interpretation would be that the part of the 5d geometry, on the boundary of whose the field theory lives, in general can have two exponents  $z$  and  $\nu$  which depends on p of Dp brane and it admits non-relativistic non-conformal symmetry and for p=3 it shows as usual the non-relativistic conformal symmetry.

The above classification is independent of whether the ten dimensional solution have a piece that admit a Sasaki-Einsteinian structure or not.

One interesting point to note that the solution eq(39) has two limits  $\beta \rightarrow 0$  and  $r_T \rightarrow 0$ . In the former limit it goes over to the original solution that we started out with i.e. the non-extremal black D4 brane solution. In the latter limit we get a solution which is the result of applying Null Melvin twist to the extremal D4 brane solution.

For completeness, let us summarize all the solutions, which may be of importance in the study of the AdS/QCD from top-down approach <sup>3</sup>

Since the  $r_T \rightarrow 0$  limit gives us an extremal solution means this is not of much important to us in the present study, however could be very interesting to understand the mesonic and glue ball spectrum in the light of study of AdS/QCD. We shall take only the  $\beta \rightarrow 0$  limit of solutions eq(39) and eq(41) for our further study.

Note that the solution eq(39) is not a rotating solution as the coordinate  $y$  is non-compact. It is also very surprising to see that the metric components depends on the angular coordinate  $\theta$ , even without rotation.

Is this generic to Dp brane solution with an even dimensional sphere, in this case  $S^4$ , transverse to the brane directions ? The answer is yes.

Just to see that let us write down the metric of a D dimensional sphere

$$d\Omega_D^2 = dx_1^2 + dx_2^2 + \dots + dx_D^2 \quad (48)$$

with a restriction  $x_1^2 + x_2^2 + \dots + x_D^2 = 1$ . Here we can have D=2d or 2d+1, even or odd dimensional sphere. For a generic D dimensional case we can have d number of independent planes so as to make rotations in each plane. for Simplicity, we shall consider the same amount of rotation in each plane.

Let us do a rotation of (say)  $x_j$  and  $x_{j+1}$  plane as

$$x_j + x_{j+1} \rightarrow e^{i\alpha y} [x_j + ix_{j+1}] \quad (49)$$

where  $y$  is the isometry direction along which we have done the boosting. This rotation means

$$dx_j^2 + dx_{j+1}^2 \rightarrow dx_j^2 + dx_{j+1}^2 + \alpha^2 (x_j^2 + x_{j+1}^2) dy^2 + 2\alpha dy (x_j dx_{j+1} - x_{j+1} dx_j) \quad (50)$$

Now it easily follows that the metric of a D=2d dimensional sphere, after the rotation

$$d\Omega_{2d}^2 = d\Omega_{2d}^2 + \alpha^2 dy^2 + 2\alpha (x_1 dx_2 - x_2 dx_1 + \dots + x_{2d-1} dx_{2d} - x_{2d} dx_{2d-1}) dy \quad (51)$$

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<sup>3</sup>As this is the only system available where one can talk of many interesting aspects like chiral symmetry breaking and confinement



whereas for an odd dimensional sphere,  $D = 2d + 1$ , the metric after rotation

$$d\Omega_{2d+1}^2 = d\Omega_{2d+1}^2 + \alpha^2(1 - x_{2d+1}^2)dy^2 + 2\alpha(x_1dx_2 - x_2dx_1 + \dots + x_{2d-1}dx_{2d} - x_{2d}dx_{2d-1})dy \quad (52)$$

The coefficient of  $g_{yy}$  of the sphere part of the metric is going to appear all over the places once we do T-duality along this direction following the rules of Null Melvin Twist. Hence the appearance of the angular variable on the metric component is a must.

The temperature of NMT resulted black hole solution eq(39) is same as the one before applying NMT to the corresponding black hole solution. Similarly, the periodicity of the extremal solution eq(41) is same as the solution before applying Null Melvin Twist. Which means the temperature of the black hole solution and the periodicity of the compact circle  $\tau$  of the extremal solution are independent of the parameter  $\beta$ .

In order to see these, first we have to compute the surface gravity

$$\kappa^2 = -\frac{1}{2}(\nabla^a \varepsilon^b)(\nabla_a \varepsilon_b) \quad (53)$$

where  $\varepsilon^b$  is a null Killing vector defined on the horizon. The temperature of the black hole is defined as  $T_H = \frac{\kappa}{2\pi}$ .

Similarly to compute the periodicity of the circle  $\tau = x^4$ , we define an analogous quantity “surface gravity” but this time without the minus sign

$$\kappa_{x^4}^2 = \frac{1}{2}(\nabla^a \epsilon^b)(\nabla_a \epsilon_b) \quad (54)$$

where  $\epsilon^b$  is a null Killing vector defined on the surface for which  $g_{x^4 x^4}$  vanishes. The inverse periodicity of this circle is defined as  $\frac{1}{2\pi R_\tau} = \frac{\kappa_{x^4}}{2\pi}$ .

Taking  $\varepsilon^b = \left(\frac{\partial}{\partial t}\right)^b$  and  $\epsilon^b = \left(\frac{\partial}{\partial x^4}\right)^b$ , we get

$$T_H = \frac{3}{4\pi} \sqrt{\frac{r_T}{R^3}}, \quad 2\pi R_\tau = \frac{4\pi}{3} \sqrt{\frac{R^3}{r_0}} \quad (55)$$

Another interesting thing is that the entropy of the black hole solution eq(39) do not depends on the twist parameter  $\beta$ . This is in agreement with [4]. Note that the solution eq(39) is written in string frame.

For completeness the solution of the black hole in Einstein frame is

$$\begin{aligned}
ds_E^2 &= \frac{1}{K^{3/4}} \left( \frac{r}{R} \right)^{9/8} \left[ -f(1+r^2\beta^2\sin^2\theta)dt^2 + (1-fr^2\beta^2\sin^2\theta)dy^2 \right. \\
&\quad \left. - 2fr^2\beta^2\sin^2\theta dtdy + K(dx^2 + dz^2 + d\tau^2) \right] + \left( \frac{R}{r} \right)^{15/8} K^{1/4} \frac{dr^2}{f} \\
&\quad + \frac{\beta^2}{K^{3/4}} \frac{R^{15/8} r_0^3}{r^{7/8}} \sin^4\theta \sin^2\varphi \cos^2\varphi (d\psi - d\chi)^2 \\
&\quad + \frac{R^{15/8} r^{1/8}}{K^{3/4}} \left[ K(d\theta^2 + \sin^2\theta d\varphi^2) + \sin^2\theta (\sin^2\varphi d\psi^2 + \cos^2\varphi d\chi^2) \right] \\
B_2 &= \frac{\beta r^2}{K} \sin^2\theta (f dt + dy) \wedge [\sin^2\varphi d\psi + \cos^2\varphi d\chi], \\
\Phi &= \Phi_0 - \frac{1}{2} \text{Log}[K], \quad F_4 = f_0 \epsilon_4, \quad K = 1 + \frac{r_T^3}{r} \beta^2 \sin^2\theta
\end{aligned} \tag{56}$$

and the extremal solution is

$$\begin{aligned}
ds_E^2 &= \left( \frac{r}{R} \right)^{9/8} \left[ - (1 + r^2\beta^2\sin^2\theta) dt^2 + (1 - r^2\beta^2\sin^2\theta) dy^2 - 2r^2\beta^2\sin^2\theta dtdy + \right. \\
&\quad \left. (dx^2 + dz^2 + f d\tau^2) \right] + \left( \frac{R}{r} \right)^{15/8} \frac{dr^2}{f(r)} + R^{15/8} r^{1/8} d\Omega_4^2, \\
B_2 &= r^2\beta\sin^2\theta (dt + dy) \wedge [\sin^2\varphi d\psi + \cos^2\varphi d\chi], \\
\Phi &= \Phi_0 = \text{Log}[g_s \left( \frac{r}{R} \right)^{3/4}], \quad F_4 = f_0 \epsilon_4.
\end{aligned} \tag{57}$$

## 4 Chiral symmetry restoration

The chiral symmetry restoration is studied [42], [43] by introducing flavor D8 and anti-D8 branes placed at two different points of the compact coordinate  $\tau$ , which has the periodicity  $\tau \sim \tau + 2\pi$ . The gauge symmetry on these branes is interpreted following Sakai-Sugimoto as the chiral symmetry  $U(N_f) \times U(N_f)$ . In the zero temperature both the branes and anti-branes get joined together so as to break this symmetry to its diagonal subgroup whereas in the finite temperature case they get separated and finally end on the horizon. This from the field theory point of view is interpreted as the chiral symmetry restoration.

Here we would like to see how do these global symmetry  $U(N_f) \times U(N_f)$  breaks to its diagonal subgroup. Probably, its difficult to define chirality in a theory that admits non-relativistic symmetry, so instead we shall interpret the breakdown of the symmetry as a breakdown of a global symmetry and chiral symmetry means we are talking about the global symmetry  $U(N_f) \times U(N_f)$ .

In order to proceed and do the calculation in this setting let us recall that the dynamics of these flavor branes are described by the DBI and CS actions, which are

$$S_{DBI} = -T_8 \int d^9 \sigma e^{-\Phi} \sqrt{-\det([g + B]_{ab} + F_{ab})} \quad (58)$$

and

$$S_{CS} = \mu_8 \int \sum_n [C^{(n)} \wedge e^B] \wedge e^{\lambda F}, \quad (59)$$

where  $[ \ ]$  denotes pull-back of the space time fields onto the world volume of the brane. In the case of interest, it is trivial to see that the CS action is not going to contribute. Hence, the only contribution comes from the DBI part. Also, in the action we have included gauge field in the action, whose interpretation from the field theory point of view is that turning on the chemical potential for the baryon number, for non-trivial electric field,  $F_{0r} = -\partial_r \mathcal{A}_0$ , [45].

The induced metric on the flavor brane, by assuming that we are only exciting the s-wave part of the mode i.e. taking the embedding to be a function of only the radial direction, for the most general solution after step 6 i.e. the eqs (35), is

$$\begin{aligned} ds^2 = & \frac{A}{K}(1 + \beta^2 DB \sin^2 \theta) dt^2 + \frac{B}{K}(1 + \beta^2 DA \sin^2 \theta) dy^2 + 2 \frac{ABD \beta^2 \sin^2 \theta}{K} dt dy \\ & + \delta_{11}(r) dx^1 dx^1 + \delta_{22}(r) dx^2 dx^2 + [\delta_{33}(r) \tau'(r)^2 + C] dr^2 + D d\theta^2 + D \sin^2 \theta d\varphi^2 \\ & + \frac{D}{K} \sin^2 \theta \sin^2 \varphi (1 + \beta^2 D \sin^2 \theta \cos^2 \varphi (A + B)) d\psi^2 \\ & + \frac{D}{K} \sin^2 \theta \cos^2 \varphi (1 + \beta^2 D \sin^2 \theta \sin^2 \varphi (A + B)) d\chi^2 \\ & - 2 \frac{\beta^2 D^2 \sin^4 \theta \sin^2 \varphi \cos^2 \varphi (A + B)}{K} d\chi d\psi, \end{aligned} \quad (60)$$

where  $\tau'(r) = \frac{d\tau}{dr}$ , whose  $\beta \rightarrow 0$  limit will give the induced metric on the flavor brane of the geometry eq(9) and taking the functions  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $\delta_{ii}$  and  $K$  appropriately can give us the induced metric for a black hole or at zero temperature, i.e. the choice of eq(38) and eq(40), respectively.

The square root of the determinant of the relevant quantity  $([g + B]_{ab} + F_{ab})$  with the induced geometry written in eq(60)

$$\sqrt{-\det([g + B]_{ab} + F_{ab})} = \frac{D^2 \sin^3 \theta \sin 2\varphi \sqrt{B\delta_{11}\delta_{22}[\mathcal{A}_0'^2 + A(C + \delta_{33}\tau'^2)]}}{2\sqrt{1 + \beta^2 D(A + B)\sin^2 \theta}}, \quad (61)$$

where we have only turned on the electric flux associated to the U(1) field strength i.e.  $F_{r0}$ . The dilaton

$$e^{-\Phi} = e^{-\Phi_0} \sqrt{1 + \beta^2 D(A + B)\sin^2 \theta}. \quad (62)$$

It means the action of the flavor brane

$$\begin{aligned} S_{DBI} &= -T_8 \int e^{-\Phi} \sqrt{-\det([g + B]_{ab} + F_{ab})} = \\ &- T_8 \int e^{-\Phi_0} \frac{D^2}{2} \sin^3 \theta \sin 2\varphi \sqrt{B\delta_{11}\delta_{22}[\mathcal{A}_0'^2 + A(C + \delta_{33}\tau'^2)]} \quad (63) \end{aligned}$$

which is independent of the parameter  $\beta$ . It just follows trivially that the action of the flavor branes before and after the Null Melvin Twists are the same. This can be confirmed by calculating the action of the flavor branes from the 10 dimensional geometry eq(9) with an electric field,  $F_{r0}$ , turned on the world volume of the flavor brane, for the functions, as written in eq(38). Similarly, evaluating eq(63) using the functions as written in eq(40) gives the same answer as that using the 10 dimensional geometry eq(9) and eq(63).

The consequence of all this is that the chiral symmetry breaking and restoration curve remains unchanged with respect to Null Melvin Twist. This can be seen very easily by taking the difference of the flavor branes evaluated on the non-zero and zero temperature backgrounds. Its because, we saw that the actions for the flavor branes remain same under Null Melvin Twist. As a result, the distance of separation between the quarks,  $L$ , remain same and hence the curve that describes the chiral symmetry restoration curve  $LT = c$ , remains same, where  $c$  is a constant and whose value is less than one.

## 5 Confinement and De-confinement transition

The confinement and de-confinement transition is a property of the bulk solution [32] and can be studied by computing the on-shell action which by AdS/CFT is related to the free energy of the system. The action of the D4 brane with the non-trivial degrees of freedom such as metric, dilaton, 2-form and 4-form antisymmetric fields,  $g_{MN}$ ,  $\Phi$ ,  $B_2$ ,  $F_4$ , respectively, is

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} [R_E - \frac{1}{2} g^{MN} \partial_M \Phi \partial_N \Phi - \frac{g_s e^{-\Phi}}{12} H_{MNP} H^{MNP} - \frac{g_s^{3/2} e^{\Phi/2}}{48} F_{MNPQ} F^{MNPQ}] \quad (64)$$

The trace of the equation motion of metric give

$$R_E - \frac{1}{2} g^{MN} \partial_M \Phi \partial_N \Phi - \frac{g_s e^{-\Phi}}{24} H_{MNP} H^{MNP} - \frac{g_s^{3/2} e^{\Phi/2}}{192} F_{MNPQ} F^{MNPQ} = 0 \quad (65)$$

Now we shall eliminate  $H_{MNP} H^{MNP}$  and write down the action in terms of other fields

$$S = -\frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} [R_E - \frac{1}{2} g^{MN} \partial_M \Phi \partial_N \Phi + \frac{g_s^{3/2} e^{\Phi/2}}{96} F_{MNPQ} F^{MNPQ}] \quad (66)$$

In order to compute the on shell action we need to know each terms that appear in the action. Various quantities of black hole solution eq(56) are

$$\begin{aligned} \sqrt{-g} &= r^{15/8} R^{15/8} \sin^3 \theta \sin \varphi \cos \varphi (r + \beta^2 r_T^3 \sin^2 \theta)^{1/4}, \\ R_E &= \frac{9(5r^3 - r_T^3)}{32r^{25/8} R^{15/8}} + \\ &\quad \beta^2 r_T^3 \frac{131r^3 - 71r_T^3 + (381r^3 + 71r_T^3) \cos 2\theta}{256r^{33/8} R^{15/8}} + \mathcal{O}(\beta^4), \\ g^{MN} \partial_M \Phi \partial_N \Phi &= \frac{9(r^3 - r_T^3)}{16r^{25/8} R^{15/8}} + \beta^2 r_T^3 \frac{39(r^3 - r_T^3) \sin^2 \theta}{64r^{33/8} R^{15/8}} + \mathcal{O}(\beta^4), \\ e^{\Phi/2} F_{MNPQ} F^{MNPQ} &= \frac{f_0^2 \sqrt{g_s} r^{1/8}}{R^{63/8} (r + \beta^2 r_T^3 \sin^2 \theta)^{1/4}} \end{aligned} \quad (67)$$

Adding all these terms with the appropriate coefficients that appear in action eq(66)

$$S_{bh} = -\frac{1}{2\kappa^2} \int_0^{\beta_H} dt \int [dy dx dz] \int_{r_T}^{R_*} dr \int_0^\pi d\theta \int_0^{\pi/2} d\varphi \int_0^{2\pi} d\psi \int_0^{2\pi} d\chi$$

$$\int_0^{2\pi\tilde{R}_\tau} d\tau \sin^3\theta \sin\varphi \cos\varphi \left[ \frac{r^2}{96R^6} (f_0^2 g_s^2 + 108R^6) + \frac{\beta^2 r_T^3}{8r^2} \left( 4r^3 - r_T^3 + (12r^3 + r_T^3) \cos 2\theta \right) + \mathcal{O}(\beta^4) \right], \quad (68)$$

where  $\beta_H = \frac{1}{T_H}$  and  $2\pi\tilde{R}_\tau$  is the periodicity of the  $\tau$  circle and the range of integration for the radial coordinate is from the horizon to a large value  $R_\star$ . For the extremal solution eq(57), various expressions are

$$\begin{aligned} \sqrt{-g} &= r^{17/8} R^{15/8} \sin^3\theta \sin\varphi \cos\varphi, \quad R_E = \frac{9(5r^3 - r_0^3)}{32r^{25/8} R^{15/8}}, \\ g^{MN} \partial_M \Phi \partial_N \Phi &= \frac{9(r^3 - r_0^3)}{16r^{25/8} R^{15/8}}, \quad e^{\Phi/2} F_{MNPQ} F^{MNPQ} = \frac{f_0^2 \sqrt{g_s}}{r^{1/8} R^{63/8}} \end{aligned} \quad (69)$$

It is easy to see that in the  $\beta \rightarrow 0$  limit, eq(67) goes over to eq(69). Note that the Ricci-scalar of eq(57) do not depend on the parameter  $\beta$ , even though the solution do. Now, the action for the extremal case

$$\begin{aligned} S_{ex} &= -\frac{1}{2\kappa^2} \int [dy dx dz] \int_0^{2\pi R_\tau} d\tau \int_{r_0}^{R_\star} dr \int_0^\pi d\theta \int_0^{\pi/2} d\varphi \int_0^{2\pi} d\psi \int_0^{2\pi} d\chi \int_0^{\tilde{\beta}} dt \\ &\quad \sin^3\theta \sin\varphi \cos\varphi \left[ \frac{r^2}{96R^6} (f_0^2 g_s^2 + 108R^6) \right], \end{aligned} \quad (70)$$

where  $\tilde{\beta}$  is the periodicity of the Euclidean time circle and  $2\pi R_\tau$  is the periodicity of the  $\tau$  circle in the extremal case.

The periodicity  $\tilde{\beta}$  is not arbitrary but is computed by taking the circumference of the Euclidean time circles be same for both the black hole and the extremal case, for a large value of the radial coordinate [32]

$$\int_0^{\tilde{\beta}} dt \sqrt{g_{tt}}|_{r=R_\star} = \int_0^{\beta_H} dt \sqrt{g_{tt}}|_{r=R_\star} \quad (71)$$

and is related to the periodicity  $\beta_H$  of the black hole as

$$\tilde{\beta} = \beta_H \left[ 1 - \left( \frac{r_T}{R_\star} \right)^3 \right]^{1/2} \left[ 1 + \frac{r_T^3}{R_\star} \beta^2 \sin^2\theta \right]^{-3/8} \quad (72)$$

and similarly for the compact  $x^4 = \tau$  circle

$$\tilde{R}_\tau = R_\tau \frac{\sqrt{1 - (r_0/R_\star)^3}}{\left[ 1 + \frac{r_T^3}{R_\star} \beta^2 \sin^2\theta \right]^{1/8}} \quad (73)$$

Now doing the appropriate integrals and keeping terms to quadratic order in  $\beta$  results

$$\begin{aligned}
S_{bh} &= -\frac{\beta_H}{2\kappa^2} V_3 2\pi^2 2\pi R_\tau \left[ \frac{f_0^2 g_s^2 + 108R^6}{216R^6} R_\star^3 - \beta^2 \frac{f_0^2 g_s^2 + 684R^6}{2160R^6} r_T^3 R_\star^2 \right. \\
&\quad \left. - \frac{f_0^2 g_s^2 + 108R^6}{432R^6} (r_0^3 + 2r_T^3) + \mathcal{O}\left(\beta^4, \frac{1}{R_\star}\right) \right] \\
S_{ex} &= -\frac{\beta_H}{2\kappa^2} V_3 2\pi^2 2\pi R_\tau \left[ \frac{f_0^2 g_s^2 + 108R^6}{216R^6} R_\star^3 - \beta^2 \frac{f_0^2 g_s^2 + 108R^6}{720R^6} r_T^3 R_\star^2 \right. \\
&\quad \left. - \frac{f_0^2 g_s^2 + 108R^6}{432R^6} (2r_0^3 + r_T^3) + \mathcal{O}\left(\beta^4, \frac{1}{R_\star}\right) \right] \tag{74}
\end{aligned}$$

The difference of the actions

$$\begin{aligned}
\Delta S = S_{bh} - S_{ex} &= -\frac{\beta_H}{2\kappa^2} V_3 2\pi^2 2\pi R_\tau \left[ (5f_0^2 g_s^2 + 540R^6)(r_0^3 - r_T^3) \right. \\
&\quad \left. + 2\beta^2 R_\star^2 r_T^3 \frac{(f_0^2 g_s^2 - 108R^6)}{2160R^6} + \mathcal{O}\left(\beta^4, \frac{1}{R_\star}\right) \right] \tag{75}
\end{aligned}$$

In the second line of eq(75) we see the appearance of a UV divergent term. This arises because we have not added the proper counter term to regulate the integrals. What we shall do is to drop this term and consider only the finite term. Of course, it is important and interesting to understand the structure of the necessary counter terms.

From the finite term of the difference of the black hole and confining actions we see that it is proportional to  $r_0^3 - r_T^3$  and is independent of the parameter  $\beta$ . Recalling the periodicity as written in eq(55) we finally get the difference as

$$\Delta S = -\frac{V_3}{2\kappa^2} 4\pi^3 R_\tau \beta_H (5f_0^2 g_s^2 + 540R^6) \left( \frac{16\pi^2 R^3}{9} \right)^3 \left[ \frac{1}{\beta_H^6} - \frac{1}{(2\pi R_\tau)^6} \right] \tag{76}$$

To see that the action go as  $\Delta S \sim N_C^2 (g_s N_c)$  in the large  $N_c$  and large  $g_s N_c$  limit, recall that<sup>4</sup>  $2\kappa^2 \sim g_s^2$ ,  $f_0^2 g_s^2 \sim R^6$ ,  $\beta_H R_\tau \sim R^3$  and  $R^3 \sim g_s N_c$  means

$$\Delta S \sim \frac{1}{g_s^2} g_s^2 N_c^2 \left( \frac{R^9 \beta_H R_\tau}{\beta_H^6} \right) \sim N_C^2 (g_s N_c) \tag{77}$$

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<sup>4</sup>In the  $2\pi l_s = 1$  unit.

In the limit  $R_\tau T_H < \frac{1}{2\pi}$ , then the extremal configuration has got lower free energy and is preferred and if  $R_\tau T_H > \frac{1}{2\pi}$  then the black hole phase is preferred. The curve that describes the confining de-confining phase transition is exactly the same as in the  $\beta \rightarrow 0$  limit.

## 6 Conclusion

We have analyzed different aspects of Null Melvin twist to D4 brane solution with the asymptotic to that of a plane wave. The resulting background at zero temperature shows confinement and at finite temperature becomes a black hole. The result of this study is showing some interesting properties that is the curve that describes the confining de confining phase transition remains unchanged so also all the periodicity at zero and non-temperature. Upon inclusion of the flavors via a bunch of coincident D8 and anti-D8 branes shows that the curve that describes the chiral symmetry breaking and restoration remains unchanged.

The vev associated to the operator which transforms bi-fundamentally under the color gauge group, for the breakdown of the chiral symmetry i.e. the chiral condensate can be calculated using the top-down prescription [47] and [48] which is given by computing the expectation value of the open Wilson line that is stretched between the D8 and anti-D8 branes and the world sheet of the Wilson line that is extended along  $r$  and  $\tau$  directions. It is interesting to know that the  $(r, \tau)$  part of the metric written in eq(57), do not depends on the parameter  $\beta$ , which means the chiral condensate also remains unchanged under Null Melvin Twist.

It has certainly become very interesting to understand more about the solution presented in [17]. Unlike the solutions [8], [9],[6] and [7], this solution has got an intrinsic exponent  $z$ , in the sense that all the coordinate invariant quantities depends on this exponent in a very non-trivial way and more importantly, the solution is not of the plane wave type.

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